NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 21, 2017

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions that follow.

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 11 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Calculus & Differential Equations, and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded for questions involving more than one answer only if **all** the correct answers are given. **There will be no partial credit**.
- Calculators are **not allowed**.

Notation

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \cdots\}$, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers.

- Let $n \in \mathbb{N}, n \geq 2$. The symbol $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) denotes the n-dimensional Euclidean space over $\mathbb{R}$ (respectively, over $\mathbb{C}$), and is assumed to be endowed with its 'usual' topology. $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will denote the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and is identified with $\mathbb{R}^{n^2}$ (respectively, $\mathbb{C}^{n^2}$) when considered as a topological space.

- The symbol $\binom{n}{r}$ will denote the standard binomial coefficient giving the number of ways of choosing r objects from a collection of n objects, where $n \geq 1$ and $0 \leq r \leq n$ are integers.

- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.

- The space of continuous real valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup-norm' metric.

- The space of continuous real valued functions on $\mathbb{R}$ which have compact support will be denoted $\mathcal{C}_c(\mathbb{R})$ and will be equipped with the 'sup-norm' metric.

- Let $1 \leq p < \infty$ and let $]a, b[\subset \mathbb{R}$ be an open interval equipped with the Lebesgue measure. The symbol $L^p(]a, b[)$ will stand for the space of measurable functions such that

  $$\int_a^b |f(t)|^p \, dt \; < \; \infty.$$

  The space $L^\infty(]a, b[)$ will stand for the space of essentially bounded functions. These spaces are equipped with their usual norms.

- The derivative of a function f is denoted by f' and the second derivative by f''.

- The symbol I will denote the identity matrix of appropriate order.

- The determinant of a square matrix A will be denoted by $\det(A)$ and its trace by $\mathrm{tr}(A)$.

- $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) will denote the group of invertible $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) with the group operation being matrix multiplication. The symbol $SL_n(\mathbb{R})$ will denote the subgroup of $GL_n(\mathbb{R})$, of matrices whose determinant is unity.

- The symbol S_n will denote the group of all permutations of n symbols $\{1, 2, \cdots, n\}$, the group operation being composition.

- The symbol $\mathbb{Z}_n$ will denote the additive group of integers modulo n.

- The symbol $\mathbb{F}_p$ will denote the field consisting of p elements, where p is a prime.

- Unless specified otherwise, all logarithms are to the base e.

Section 1: Algebra

1.1 Let G be a group. Which of the following statements are true?
a. Let H and K be subgroups of G of orders 3 and 5 respectively. Then $H \cap K = \{e\}$, where e is the identity element of G.
b. If G is an abelian group of odd order, then $\varphi(x) = x^2$ is an automorphism of G.
c. If G has exactly one element of order 2, then this element belongs to the centre of G.

1.2 Let $n \in \mathbb{N}, n \geq 2$. Which of the following statements are true?
a. Any finite group G of order n is isomorphic to a subgroup of $GL_n(\mathbb{R})$.
b. The group $\mathbb{Z}_n$ is isomorphic to a subgroup of $GL_2(\mathbb{R})$.
c. The group $\mathbb{Z}_{12}$ is isomorphic to a subgroup of S_7.

1.3 Which of the following statements are true?
a. The matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

are conjugate in $GL_2(\mathbb{R})$.
b. The matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

are conjugate in $SL_2(\mathbb{R})$.
c. The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

are conjugate in $GL_2(\mathbb{R})$.

1.4 Let p be an odd prime. Find the number of non-zero squares in $\mathbb{F}_p$.

1.5 Find a generator of $\mathbb{F}_7^\times$, the multiplicative group of non-zero elements of $\mathbb{F}_7$.

1.6 The characteristic polynomial of a matrix $A \in \mathbb{M}_5(\mathbb{R})$ is given by $x^5 + \alpha x^4 + \beta x^3$, where α and β are non-zero real numbers. What are the possible values of the rank of A?

1.7 Let $A \in \mathbb{M}_3(\mathbb{R})$ be a symmetric matrix whose eigenvalues are $1, 1$ and 3. Express A^{-1} in the form $\alpha I + \beta A$, where $\alpha, \beta \in \mathbb{R}$.

1.8 Let $A \in \mathbb{M}_n(\mathbb{R}), n \geq 2$. Which of the following statements are true?
a. If $A^{2n} = 0$, then $A^n = 0$.
b. If $A^2 = I$, then $A = \pm I$.
c. If $A^{2n} = I$, then $A^n = \pm I$.

1.9 Which of the following statements are true?
a. There does not exist a non-diagonal matrix $A \in \mathbb{M}_2(\mathbb{R})$ such that $A^3 = I$.
b. There exists a non-diagonal matrix $A \in \mathbb{M}_2(\mathbb{R})$ which is diagonalizable over $\mathbb{R}$ and which is such that $A^3 = I$.
c. There exists a non-diagonal matrix $A \in \mathbb{M}_2(\mathbb{R})$ such that $A^3 = I$ and such that $\mathrm{tr}(A) = -1$.

1.10 Let $n \geq 2$ and let W be the subspace of $\mathbb{M}_n(\mathbb{R})$ consisting of all matrices whose trace is zero. If $A = (a_{ij})$ and $B = (b_{ij})$, for $1 \leq i, j \leq n$, are elements in $\mathbb{M}_n(\mathbb{R})$, define their inner-product by

$$(A, B) = \sum_{i,j=1}^{n} a_{ij} b_{ij}.$$

Identify the subspace $W^\perp$ of elements orthogonal to the subspace W.

Section 2: Analysis

2.1 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $\alpha = \liminf_{n\to\infty} x_n$. Which of the following statements are true?
a. For every $\varepsilon > 0$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \leq \alpha + \varepsilon$ for all $k \in \mathbb{N}$.
b. For every $\varepsilon > 0$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \leq \alpha - \varepsilon$ for all $k \in \mathbb{N}$.
c. There exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to \alpha$ as $k \to \infty$.

2.2 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ is divergent. Which of the following series are convergent?
a.
$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}.$$
b.
$$\sum_{n=1}^{\infty} \frac{a_n}{1+n^2a_n}.$$
c.
$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}.$$

2.3 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ is convergent. Which of the following series are convergent?
a.
$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}.$$
b.
$$\sum_{n=1}^{\infty} \frac{a_n^{\frac{1}{4}}}{n^{\frac{4}{5}}}.$$
c.
$$\sum_{n=1}^{\infty} na_n \sin\frac{1}{n}.$$

2.4 Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. It is said to be *lower semi-continuous* (respectively *upper semi-continuous*) if the set $f^{-1}(]-\infty,\alpha])$ (respectively, the set $f^{-1}([\alpha,\infty[)$) is closed for every $\alpha \in \mathbb{R}$. Let f and g be two real valued functions defined on $\mathbb{R}$. Which of the following statements are true?
a. If f and g are continuous, then $\max\{f,g\}$ is continuous.
b. If f and g are lower semi-continuous, then $\max\{f,g\}$ is lower semi-continuous.
c. If f and g are upper semi-continuous, then $\max\{f,g\}$ is upper semi-continuous.

2.5 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Which of the following statements are true?
a. If f is continuously differentiable, then f is uniformly continuous.
b. If f has compact support, then f is uniformly continuous.
c. If $\lim_{|x|\to\infty} |f(x)| = 0$, then f is uniformly continuous.

2.6 Let $f :]0, 2[\to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \in]0, 2[\cap \mathbb{Q}, \\ 2x - 1, & \text{if } x \in]0, 2[\setminus \mathbb{Q}. \end{cases}$$

Check for the points of differentiability of f and evaluate the derivative at those points.

2.7 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous real valued functions defined on $\mathbb{R}$ which converges pointwise to a continuous real valued function f. Which of the following statements are true?
a. If $0 \leq f_n \leq f$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\, dt = \int_{-\infty}^{\infty} f(t)\, dt.$$

b. If $|f_n(t)| \leq |\sin t|$ for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\, dt = \int_{-\infty}^{\infty} f(t)\, dt.$$

c. If $|f_n(t)| \leq e^t$ for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$, then for all $a, b \in \mathbb{R}, a < b$,

$$\lim_{n\to\infty} \int_{a}^{b} f_n(t)\, dt = \int_{a}^{b} f(t)\, dt.$$

2.8 Which of the following statements are true?
a. The following series is uniformly convergent over $[-1, 1]$:

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}.$$

b.

$$\lim_{n\to\infty} \int_{\frac{\pi}{2}}^{\pi} \frac{\sin nx}{nx^5}\, dx = \pi.$$

c. Define, for $x \in \mathbb{R}$,

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx^2}{1+n^3}.$$

Then f is a continuously differentiable function.

2.9 Write down the Laurent series expansion of the function

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

in the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$.

2.10 Which of the following statements are true?
a. There exists a non-constant entire function which is bounded on the real and imaginary axes of $\mathbb{C}$.
b. The ring of analytic functions on the open unit disc of $\mathbb{C}$ (with respect to the operations of pointwise addition and pointwise multiplication) is an integral domain.
c. There exists an entire function f such that $f(0) = 1$ and such that $|f(z)| \leq \frac{1}{|z|}$ for all $|z| \geq 5$.

Section 3: Topology

3.1 Let (X, d) be a metric space and let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be arbitrary Cauchy sequences in X. Which of the following statements are true?
a. The sequence $\{d(x_n, y_n)\}$ converges as $n \to \infty$.
b. The sequence $\{d(x_n, y_n)\}$ converges as $n \to \infty$ only if $X$is complete.
c. No conclusion can be drawn about the convergence of $\{d(x_n, y_n)\}$.

3.2 Which of the following statements are true?
a. Let X be a set equipped with two topologies τ_1 and τ_2. Assume that any given sequence in X converges with respect to the topology τ_1 if, and only if, it also converges with respect to the topology τ_2. Then $\tau_1 = \tau_2$.
b. Let (X, τ_1) and (Y, τ_2) be two topological spaces and let $f : X \to Y$ be a given map. Then f is continuous if, and only if, given any sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y.
c. Let (X, τ) be a compact topological space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Then, it has a convergent subsequence.

3.3 Which of the following statements are true?
a. Let $n \geq 2$. The subset of nilpotent matrices in $\mathbb{M}_n(\mathbb{C})$ is closed in $\mathbb{M}_n(\mathbb{C})$.
b. Let $n \geq 2$. The set of all matrices in $\mathbb{M}_n(\mathbb{C})$ which represent orthogonal projections is closed in $\mathbb{M}_n(\mathbb{C})$.
c. The set of all matrices in $\mathbb{M}_2(\mathbb{R})$ such that both of their eigenvalues are purely imaginary, is closed in $\mathbb{M}_2(\mathbb{R})$.

3.4 Which of the following sets are dense?
a. The set of all numbers of the form $\frac{m}{2^n}$ where $0 \leq m \leq 2^n$ and $n \in \mathbb{N}$, in the space $[0, 1]$.
b. The set of all polynomial functions in the space $L^1(]0, 1[)$.
c. The linear span of the family $\{\sin nt\}_{n=1}^{\infty}$ in the space $L^2(]-\pi, \pi[)$.

3.5 Let $n \geq 2$. Which of the following subsets are nowhere dense in $\mathbb{M}_n(\mathbb{R})$?
a. The set $GL_n(\mathbb{R})$.
b. The set of all matrices whose trace is zero.
c. The set of all singular matrices.

3.6 Which of the following topological spaces are separable?
a. Any real Banach space which admits a Schauder basis $\{u_n\}_{n=1}^{\infty}$.
b. The space $\mathcal{C}[0, 1]$.
c. The space $L^p(]0, 1[)$, where $1 \leq p \leq \infty$.

3.7 Which of the following sets are connected?
a. The set of all points in the plane with at least one coordinate irrational.
b. An infinite set X with the topology τ given by

$$\tau = \{X, \emptyset\} \cup \{A \subset X \mid X \backslash A \text{ is a finite set}\}.$$

c. The set

$$K = \{f \in \mathcal{C}[0, 1] \mid \int_0^{\frac{1}{2}} f(t)\, dt - \int_{\frac{1}{2}}^1 f(t)\, dt = 1\}.$$

3.8 Which of the following statements are true?
a. There exists a continuous bijection $f : [0,1] \to [0,1] \times [0,1]$.
b. There exists a continuous map $f : S^1 \to \mathbb{R}$ which is injective, where S^1 stands for the unit circle in the plane.
c. There exists a continuous map $f : [0,1] \to SL_2(\mathbb{R})$ which is surjective.

3.9 Which of the following statements are true?
a. Let $g \in \mathcal{C}[0,1]$ be fixed. Then the set

$$A = \{f \in \mathcal{C}[0,1] \mid \int_0^1 f(t)g(t)\ dt = 0\}$$

is closed in $\mathcal{C}[0,1]$.
b. Let $g \in \mathcal{C}_c(\mathbb{R})$, be fixed. Then the set

$$A = \{f \in \mathcal{C}_c(\mathbb{R}) \mid \int_{-\infty}^{\infty} f(t)g(t)\ dt = 0\}$$

is closed in $\mathcal{C}_c(\mathbb{R})$.
c. Let $g \in L^2(\mathbb{R})$ be fixed. Then the set

$$A = \{f \in L^2(\mathbb{R}) \mid \int_{-\infty}^{\infty} f(t)g(t)\ dt = 0\}$$

is closed in $L^2(\mathbb{R})$.

3.10 Which of the following statements are true?
a. Let X be a compact topological space and let $\mathcal{F}$ be a family of real valued functions defined on X with the following properties:
(i) If $f, g \in \mathcal{F}$, then $fg \in \mathcal{F}$, where $(fg)(x) = f(x)g(x)$ for all $x \in X$.
(ii) For every $x \in X$, there exists an open neighbourhood $U(x)$ of x and a function $f \in \mathcal{F}$ such that the restiction of f to $U(x)$ is identically zero.
Then the function which is identically zero on all of X belongs to $\mathcal{F}$.
b. Let

$$X = \{f : [0,1] \to [0,1] \mid |f(t) - f(s)| \le |t - s| \text{ for all } s, t \in [0,1]\}.$$

Define

$$d(f,g) = \max_{t \in [0,1]} |f(t) - g(t)|$$

for $f, g \in X$. Then (X, d) is a compact metric space.
c. Let $\{f_i\}_{i \in I}$ be a collection of functions in $\mathcal{C}[0,1]$ such that given any finite subfamily of functions, its members vanish at some common point (which depends on that subfamily). Then there exists $x_0 \in [0,1]$ such that $f_i(x_0) = 0$ for all $i \in I$.

Section 4: Calculus & Differential Equations

4.1 Let $x > 1$. Define

$$F(x) \;=\; \int_{x^2}^{x^3} \tan(xy^2)\; dy.$$

Differentiate F with respect to x.

4.2 Evaluate:

$$\int_{-\infty}^{\infty} e^{-2x^2}\; dx.$$

4.3 Let $\mathbf{n}(x, y, z)$ denote the unit outer normal vector on the surface S of the cylinder $x^2 + y^2 \leq 4,\; 0 \leq z \leq 3$. Compute

$$\int_S \mathbf{v}.\mathbf{n}\; dS$$

where $\mathbf{v}(x, y, z) = xz\mathbf{i} + 2yz\mathbf{j} + 3xy\mathbf{k}$.

4.4 Evaluate the line integral $\int_C Pdx + Qdy$, where C is the circle centered at the origin and of radius $a > 0$ (described in the counter-clockwise sense) in the plane and

$$P(x, y) \;=\; \frac{-y}{x^2 + y^2},\;\; Q(x, y) \;=\; \frac{x}{x^2 + y^2}.$$

4.5 Let Ω be a bounded open subset of $\mathbb{R}^3$ and let $\partial\Omega$ denote its boundary. Given sufficiently smooth real valued fucntions u and v on $\overline{\Omega}$, let $\frac{\partial u}{\partial n}$ and $\frac{\partial v}{\partial n}$ denote the outer normal derivatives of u and v respectively on $\partial\Omega$. Fill in the blank in the following identity:

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right)\; dS \;=\; \int_\Omega (\cdots\cdots\cdots)\; dx\; dy\; dz.$$

4.6 Find the maximum value of $x^2 + xy$ subject to the condition $x^2 + y^2 \leq 1$.

4.7 Interchange the order of integration:

$$\int_{-1}^{2} \int_{-x}^{2-x^2} f(x, y)\; dy\; dx.$$

4.8 Find all the non-trivial solutions (λ, u) (*i.e.* $u \not\equiv 0$), of the boundary value problem:

$$-u''(x) \;=\; \lambda u(x), 0 < x < 1, \text{ and } u(0) = u'(1) = 0.$$

4.9 Consider the initial value problem: $u'(t) = Au(t), t > 0$, and $u(0) = u_0$, where u_0 is a given vector in $\mathbb{R}^2$ and

$$A \;=\; \begin{bmatrix} 1 & -2 \\ 1 & a \end{bmatrix}.$$

Find the range of values of a such that $|u(t)| \to 0$ as $t \to \infty$.

4.10 Let $u(x,t)$ be the solution of the wave equation:

$$\left.\begin{array}{rcl} \frac{\partial^2 u}{\partial t^2} & = & \frac{\partial^2 u}{\partial x^2},\ x \in \mathbb{R}, t > 0, \\ u(x,0) & = & u_0(x),\ \ x \in \mathbb{R}, \\ u_t(x,0) & = & 0,\ \ x \in \mathbb{R}. \end{array}\right\}$$

Let $u_0(x)$ be the function defined by

$$u_0(x) \ = \ \begin{cases} 1, & \text{if } |x| < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $u(x,1)$ at all points $x \in \mathbb{R}$ where it is continuous.

Section 5: Miscellaneous

5.1 Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Evaluate:

$$\sum_{k=0}^{n} \binom{n}{k} \sin\left(x + \frac{k\pi}{2}\right).$$

5.2 Let $n \in \mathbb{N}, n \geq 2$. Let $x_1, \cdots, x_n \in]0, \pi[$. Set $x = (x_1 + \cdots + x_n)/n$. Which of the following statements are true?

a.

$$\Pi_{k=1}^{n} \sin x_k \ \geq \ \sin^n x.$$

b.

$$\Pi_{k=1}^{n} \sin x_k \ \leq \ \sin^n x.$$

c. Neither (a) nor (b) is necessarily true.

5.3 Which of the following sets are convex?

a.

$$\{(x,y) \in \mathbb{R}^2 \mid xy \geq 1, x \geq 0, y \geq 0\}.$$

b.

$$\{(x,y) \in \mathbb{R}^2 \mid |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} \leq 1\}.$$

c.

$$\{(x,y) \in \mathbb{R}^2 \mid y \geq x^2\}.$$

5.4 Find the area of the circle got by intersecting the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x + y + z = 1$.

5.5 Let $n \in \mathbb{N}, n \geq 3$. Find the area of the polygon with one vertex at $z = 1$ and whose other vertices are situated at the roots of the polynomial

$$1 + z + z^2 + \cdots + z^{n-1}$$

in the complex plane.

5.6 Find the maximum value of $3x + 2y$ subject to the conditions:

$$2x + 3y \geq 6, \ y - x \leq 2, \ 0 \leq x \leq 3, \ y \geq 0$$

.

5.7 A committee of six members is formed from a group of 7 men and 4 women. What is the probability that the commitee contains

a. exactly two women?

b. at least two women?

5.8 Find the sum of the infinite series:

$$\frac{1}{2.3.4} + \frac{1}{4.5.6} + \frac{1}{6.7.8} + \cdots$$

5.9 Find the remainder when 8^{130} is divided by 13.

5.10 Let $a_i \in \mathbb{R}, 1 \leq i \leq 4$. Evaluate:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 \end{vmatrix}.$$