NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 28, 2012

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions that follow.

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 12 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded for questions involving more than one answer only if **all** the correct answers are given. **There will be no partial credit**.
- Calculators are **not allowed**.

Notation

- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers.

- $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) denotes the n-dimensional Euclidean space over $\mathbb{R}$ (respectively, over $\mathbb{C}$), and is assumed to be endowed with its 'usual' topology. $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will denote the set of all $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and is identified with $\mathbb{R}^{n^2}$ (respectively, $\mathbb{C}^{n^2}$) when considered as a topological space.

- The symbol $\mathbb{Z}_n$ will denote the ring of integers modulo n.

- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.

- The space of continuous real-valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup-norm' metric. The space of continuously differentiable real-valued functions on $[a, b]$ is denoted by $\mathcal{C}^1[a, b]$.The symbol $\mathcal{C}^\infty$ will denote the corresponding space of infinitely differentiable functions.

- The derivative of a function f is denoted by f' and the second derivative by f''.

- The symbol I will denote the identity matrix of appropriate order.

- The determinant of a square matrix A will be denoted by $\det(A)$ and its trace by $\mathrm{tr}(A)$.

- $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) will denote the group of invertible $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) with the group operation being matrix multiplication.

Section 1: Algebra

1.1 Which of the following are subgroups of $GL_3(\mathbb{C})$?

a.

$$H = \{A \in \mathbb{M}_3(\mathbb{C}) \mid \det(A) = 2^l, l \in \mathbb{Z}\}.$$

b.

$$H = \left\{ \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

c.

$$H = \left\{ \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid \alpha \in \mathbb{C} \right\}.$$

1.2 Let S_7 denote the symmetric group of all permutations of the symbols $\{1, 2, 3, 4, 5, 6, 7\}$. Pick out the true statements:
a. S_7 has an element of order 10;
b. S_7 has an element of order 15;
c. the order of any element of S_7 is at most 12.

1.3 Let $\mathcal{C}(\mathbb{R})$ denote the ring of all continuous real-valued functions on $\mathbb{R}$, with the operations of pointwise addition and pointwise multiplication. Which of the following form an ideal in this ring?
a. The set of all $\mathcal{C}^\infty$ functions with compact support.
b. The set of all continuous functions with compact support.
c. The set of all continuous functions which vanish at infinity, *i.e.* functions f such that $\lim_{|x|\to\infty} f(x) = 0$.

1.4 Find the number of non-zero elements in the field $\mathbb{Z}_p$, where p is an odd prime number, which are squares, *i.e.* of the form $m^2, m \in \mathbb{Z}_p, m \neq 0$.

1.5 Find the inverse in $\mathbb{Z}_5$ of the following matrix:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

1.6 Let $\mathcal{P}_3$ denote the (real) vector space of all polynomials (in one variable), with real coefficients and of degree less than, or equal to, 3, equipped with the standard basis $\{1, x, x^2, x^3\}$. Write down the matrix (with respect to this basis) of the linear transformation

$$L(p) = p'' - 2p' + p, \ p \in \mathcal{P}_3.$$

1.7 Find the unique polynomial $p \in \mathcal{P}_3$ (see Question 1.6 above) such that

$$p'' - 2p' + p \;=\; x^3.$$

1.8 Let $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{R}), n \geq 3$. Let $B = (b_{ij})$ be the matrix of its cofactors, *i.e.* b_{ij} is the cofactor of the entry a_{ij} in A. What is the rank of B when
a. the rank of A is n?
b. the rank of A is less than, or equal to, $n - 2$?

1.9 Let $A \in \mathbb{M}_3(\mathbb{R})$ which is **not** a diagonal matrix. Pick out the cases when A is diagonalizable over $\mathbb{R}$:
a. when $A^2 = A$;
b. when $(A - 3I)^2 = 0$;
c. when $A^2 + I = 0$.

1.10 Let $A \in \mathbb{M}_3(\mathbb{R})$ which is **not** a diagonal matrix. Let p be a polynomial (in one variable), with real coefficients and of degree 3 such that $p(A) = 0$. Pick out the true statements:
a. $p = cp_A$ where $c \in \mathbb{R}$ and p_A is the characteristic polynomial of A;
b. if p has a complex root (*i.e.* a root with non-zero imaginary part), then $p = cp_A$, with c and p_A as above;
c. if p has a complex root, then A is diagonalizable over $\mathbb{C}$.

Section 2: Analysis

2.1 Which of the following statements are true?
a. Let $\{a_{mn}\}$, $m,n \in \mathbb{N}$, be an arbitrary double sequence of real numbers. Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^3 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}^3.$$

b. Let $\{a_{mn}\}$, $m,n \in \mathbb{N}$, be an arbitrary double sequence of real numbers. Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}^2.$$

c. Let $\{a_{mn}\}$, $m,n \in \mathbb{N}$, be a double sequence of real numbers such that $|a_{mn}| \le \sqrt{m/n}$ for all $m,n \in \mathbb{N}$. Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_{mn}}{m^2 n} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_{mn}}{m^2 n}.$$

2.2 Let $f \in \mathcal{C}[-1,1]$. Evaluate:

$$\lim_{h\to 0} \frac{1}{h}\int_{-h}^{h} f(t)\ dt.$$

2.3 Let $f \in \mathcal{C}^1[-1,1]$. Evaluate:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f'\left(\frac{k}{3n}\right).$$

2.4 Let $f \in \mathcal{C}[-\pi,\pi]$. Evaluate:
a.

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\cos nt\ dt;$$

b.

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\cos^2 nt\ dt;$$

2.5 In each of the following cases, examine whether the given sequence (or series) of functions converges uniformly over the given domain:
a.

$$f_n(x) = \frac{nx}{1+nx},\ x \in]0,\infty[;$$

b.

$$\sum_{n=1}^{\infty} \frac{n\sin nx}{e^n},\ x \in [0,\pi];$$

c.

$$f_n(x) = \frac{x^n}{1+x^n},\ x \in [0,2].$$

2.6 Compute $F'(x)$ where

$$F(x) = \int_{-x}^{x} \frac{1-e^{-xy}}{y}\ dy,\ x > 0.$$

2.7 Let $a > 0$ and let $k \in \mathbb{N}$. Evaluate:

$$\lim_{n\to\infty} a^{-nk}\Pi_{j=1}^{k}\left(a+\frac{j}{n}\right)^{n}.$$

2.8 Write down the power series expansion of the function $f(z) = 1/z^2$ about the point $z = 2$.

2.9 Let C be the circle $|z + 2| = 3$ described in the anti-clockwise (*i.e.* positive) sense in the complex plane. Evaluate:

$$\int_C \frac{dz}{z^3(z+4)}.$$

2.10 Which of the following statements are true?
a. There exists an entire function $f : \mathbb{C} \to \mathbb{C}$ which takes only real values and is such that f(0) = 0 and $f(1) = 1$.
b. There exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f(n + \frac{1}{n}) = 0$ for all $n \in \mathbb{N}$.
c. There exists an entire function $f : \mathbb{C} \to \mathbb{C}$ which is onto and which is such that $f(1/n) = 0$ for all $n \in \mathbb{N}$.

Section 3: Topology

3.1 Let A and B be subsets of $\mathbb{R}^n$. Define

$$A + B \;=\; \{x + y \mid x \in A, y \in B\}.$$

Pick out the true statements:
a. if A and B are closed sets, then $A + B$ is a closed set;
b. if A is an open set and if B is a closed set, then $A + B$ is an open set;
c. if A and B are compact sets, then so is $A + B$.

3.2 Let X and Y be metric spaces and let $f : X \to Y$ be a mapping. Pick out the true statements:
a. if f is uniformly continuous, then the image of every Cauchy sequence in X is a Cauchy sequence in Y;
b. if X is complete and if f is continuous, then the image of every Cauchy sequence in X is a Cauchy sequence in Y;
c. if Y is complete and if f is continuous, then the image of every Cauchy sequence in X is a Cauchy sequence in Y;

3.3 Which of the following statements are true?
a. If A is a dense subset of a topological space X, then $X\backslash A$ is nowhere dense in X.
b. If A is a nowhere dense subset of a topological space X, then $X\backslash A$ is dense in X.
c. The set $\mathbb{R}$, identified with the x-axis in $\mathbb{R}^2$, is nowhere dense in $\mathbb{R}^2$.

3.4 Which of the following metric spaces are separable?
a. The space $\mathcal{C}[0, 1]$, with the usual 'sup-norm' metric.
b. The space ℓ_1 of all absolutely convergent real sequences, with the metric

$$d_1(\{a_i\}, \{b_i\}) \;=\; \sum_{i=1}^{\infty} |a_i - b_i|.$$

c. The space ℓ_∞ of all bounded real sequences, with the metric

$$d_\infty(\{a_i\}, \{b_i\}) \;=\; \sup_{1 \le i < \infty} |a_i - b_i|.$$

3.5 Which of the following sets are compact?
a. The closed unit ball centred at 0 and of radius 1 of ℓ_1 (see Question 3.4(b) above).
b. The set of all unitary matrices in $\mathbb{M}_2(\mathbb{C})$.
c. The set of all matrices in $\mathbb{M}_2(\mathbb{C})$ with determinant equal to unity.

3.6 Which of the following sets are connected?
a. The set $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ in $\mathbb{R}^2$.
b. The set of all symmetric matrices in $\mathbb{M}_n(\mathbb{R})$.
c. The set of all orthogonal matrices in $\mathbb{M}_n(\mathbb{R})$.

3.7 Which of the following metric spaces are complete?
a. The space of all continuous real-valued functions on $\mathbb{R}$ with compact support, with the usual 'sup-norm' metric.
b. The space $\mathcal{C}[0,1]$ with the metric

$$d_1(f,g) \;=\; \int_0^1 |f(t)-g(t)|\,dt.$$

c. The space $\mathcal{C}^1[0,1]$ with the metric

$$d(f,g) \;=\; \max_{t\in[0,1]} |f(t)-g(t)|.$$

3.8 Let $X_j = \mathcal{C}[0,1]$ with the metric d_1 (see Question 3.7(b) above) when $j=1$, the metric

$$d_2(f,g) \;=\; \left(\int_0^1 |f(t)-g(t)|^2\,dt\right)^{\frac{1}{2}}$$

when $j=2$ and the usual 'sup-norm' metric when $j=3$. Let $\mathrm{id} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be the identity map. Pick out the true statements:
a. $\mathrm{id} : X_2 \to X_1$ is continuous;
b. $\mathrm{id} : X_1 \to X_3$ is continuous;
c. $\mathrm{id} : X_3 \to X_2$ is continuous.

3.9 Which of the following statements are true?
a. Consider the subspace $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ of $\mathbb{R}^2$. Then, there exists a continuous function $f : S^1 \to \mathbb{R}$ which is onto.
b. There exists a continuous function $f : S^1 \to \mathbb{R}$ which is one-one.
c. Let

$$X \;=\; \{A = (a_{ij}) \in \mathbb{M}_2(\mathbb{R}) \mid \mathrm{tr}(A) = 0 \text{ and } |a_{ij}| \le 2 \text{ for all } 1 \le i,j \le 2\}.$$

Let $Y = \{\det(A) \mid A \in X\} \subset \mathbb{R}$. Then, there exist $\alpha < 0$ and $\beta > 0$ such that $Y = [\alpha,\beta]$.

3.10 Let $S^1 \subset \mathbb{R}^2$ be as in Question 3.9(a) above. Let

$$D \;=\; \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\} \text{ and } E \;=\; \{(x,y) \in \mathbb{R}^2 \mid 2x^2+3y^2 \le 1\}$$

be also considered as subspaces of $\mathbb{R}^2$. Which of the following statements are true?
a. If $f : D \to S^1$ is a continuous mapping, then there exists $x \in S^1$ such that $f(x) = x$.
b. If $f : S^1 \to S^1$ is a continuous mapping, then there exists $x \in S^1$ such that $f(x) = x$.
c. If $f : E \to E$ is a continuous mapping, then there exists $x \in E$ such that $f(x) = x$.

Section 4: Applied Mathematics

4.1 Find the family of orthogonal trajectories of the family of curves $y = cx^2$.

4.2 Let $f \in \mathcal{C}[0,1]$ be given. Consider the problem: find a curve u such that $u(0) = u(1) = 0$ which minimizes the functional

$$J(v) \;=\; \frac{1}{2}\int_0^1 (v'(x))^2\,dx - \int_0^1 f(x)v(x)\,dx$$

over all admissible curves v. Write down the boundary value problem (Euler-Lagrange equation) satisfied by the solution u.

4.3 Let $\omega \in \mathbb{R}$ be a constant. Solve:

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t) - \omega y(t), \quad t > 0 \\ \frac{dy(t)}{dt} &= \omega x(t) + y(t), \quad t > 0 \\ x(0) &= y(0) = 1. \end{aligned}$$

4.4 Let $b \in R$ be a constant. Solve:

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) + b\frac{\partial u}{\partial x}(x,t) &= 0, \qquad x \in \mathbb{R},\ t > 0 \\ u(x,0) &= x^2. \end{aligned}$$

4.5 Let v be a smooth harmonic function on $\mathbb{R}^n$. If $r^2 = \sum_{i=1}^n |x_i|^2$, where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and if v is a radial function, *i.e.* $v(x) = v(r)$, write down the ordinary differential equation satisfied by v.

4.6 Let a and b be positive constants. Let y satisfy

$$\begin{aligned} y''(t) + ay'(t) + by(t) &= 0, \quad t > 0, \\ y(0) = 1 \text{ and } y'(0) &= 1. \end{aligned}$$

Write down the Laplace transform of y.

4.7 Write down the Newton-Raphson iteration scheme to find $1/\sqrt{a}$, where $a > 0$, by solving the equation $x^{-2} - a = 0$.

4.8 Let $B \in \mathbb{M}_n(\mathbb{R})$ and let $b \in \mathbb{R}^n$ be a given fixed vector. Consider the iteration scheme

$$x_{n+1} \;=\; Bx_n + b,\ x_0 \text{ given.}$$

Pick out the true statements:
a. the scheme is always convergent for any initial vector x_0.
b. if the scheme is always convergent for any initial vector x_0, then $I - B$ is invertible.
c. if the scheme is always convergent for any initial vector x_0, then every eigenvalue λ of B satisfies $|\lambda| < 1$.

4.9 Let

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 \\ 1 & 1 & 0 & 3 \\ -1 & 1 & 1 & -1 \\ 0 & -3 & 1 & 1 \end{bmatrix}.$$

Pick out the smallest disc in the complex plane containing all the eigenvalues of A from amongst the following:
a. $|z-1| \leq 7$;
b. $|z-1| \leq 6$;
c. $|z-1| \leq 4$.

4.10 Solve: maximize $z = 7x + 5y$ such that $x \geq 0, y \geq 0$ and

$$\begin{aligned} x + 2y &\leq 6 \\ 4x + 3y &\leq 12. \end{aligned}$$

Section 5: Miscellaneous

5.1 Which of the following sets are countable?
a. The set of all sequences of non-negative integers.
b. The set of all sequences of non-negative integers with only a finite number of non-zero terms.
c. The set of all roots of all monic polynomials in one variable with rational coefficients.

5.2 A magic square of order N is an $N \times N$ matrix with positive integral entries such that the elements of every row, every column and the two diagonals all add up to the same number. If a magic square is filled with numbers in arithmetic progression starting with $a \in \mathbb{N}$ and common difference $d \in \mathbb{N}$, what is the value of this common sum?

5.3 A committee consists of n members and a group photograph is to be taken by seating them in a row. If two particular members do not get along with each other, in how many ways can the committee members be seated so that these two are never adjacent to each other?

5.4 Let $n \geq 2$ and let D_n be the number of permutations of $\{1, 2, \cdots, n\}$ which leave no symbol fixed. (For example: $D_2 = 1$). Write down an expression for D_n in terms of $D_k, 2 \leq k \leq n-1$.

5.5 Five letters are addressed to five different persons and the corresponding envelopes are prepared. The letters are put into the envelopes at random. What is the probability that no letter is in its proper envelope?

5.6 Which of the following statements are true?
a. The 9-th power of any positive integer is of the form $19m$ or $19m \pm 1$.
b. For any positive integer n, the number $n^{13} - n$ is divisible by 2730.
c. The number $18! + 1$ is divisible by 437.

5.7 Let $a_i, 1 \leq i \leq n$ be non-negative real numbers. Let S denote their sum. Pick out the true statements:
a.
$$\Pi_{k=1}^{n}(1 + a_k) \geq 1 + S;$$
b.
$$\Pi_{k=1}^{n}(1 + a_k) \leq 1 + \frac{S}{1!} + \frac{S^2}{2!} + \cdots + \frac{S^n}{n!};$$
c.
$$\Pi_{k=1}^{n}(1 + a_k) \leq \frac{1}{1 - S}, \text{ if } S < 1$$
.

5.8 Consider the Fibonacci sequence $\{a_n\}$ defined by

$$a_0 = 0,\ a_1 = 1,\ a_{n+1} = a_n + a_{n-1},\ n \geq 1.$$

Write down its enumerating function, *i.e.* the function with the formal power series expansion $\sum_{n=0}^{\infty} a_n x^n$.

5.9 Find the lengths of the semi-axes of the ellipse whose equation is given by

$$5x^2 - 8xy + 5y^2 = 1.$$

5.10 Let $x_0 = a$ and $x_1 = b$. Define

$$x_{n+1} = \left(1 - \frac{1}{2n}\right) x_n + \frac{1}{2n} x_{n-1},\ n \geq 1.$$

Find $\lim_{n\to\infty} x_n$.