

**NATIONAL BOARD FOR HIGHER MATHEMATICS**

**Research Scholarships Screening Test**

**Saturday, January 23, 2010**

**Time Allowed: 150 Minutes**

**Maximum Marks: 40**

**Please read, carefully, the instructions on the following page**

## INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 11 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. **There will be no partial credit.**
- $\mathbb{N}$  denotes the set of natural numbers,  $\mathbb{Z}$  - the integers,  $\mathbb{Q}$  - the rationals,  $\mathbb{R}$  - the reals and  $\mathbb{C}$  - the field of complex numbers.  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space, which is assumed to be endowed with its 'usual' topology. The symbol  $\mathbb{Z}_n$  will denote the ring of integers modulo  $n$ . The symbol  $]a, b[$  will stand for the open interval  $\{x \in \mathbb{R} \mid a < x < b\}$  while  $[a, b]$  will stand for the corresponding closed interval;  $[a, b[$  and  $]a, b]$  will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol  $I$  will denote the identity matrix of appropriate order. The space of continuous real valued functions on an interval  $[a, b]$  is denoted by  $\mathcal{C}[a, b]$  and is endowed with its usual 'sup' norm.

## Section 1: Algebra

1.1 Solve the equation

$$x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$$

given that two of its roots are equal in magnitude but opposite in sign.

1.2 Let  $G$  be a group. A subgroup  $H$  of  $G$  is called *characteristic* if  $\varphi(H) \subset H$  for all automorphisms  $\varphi$  of  $G$ . Pick out the true statement(s):

- (a) Every characteristic subgroup is normal.
- (b) Every normal subgroup is characteristic.
- (c) If  $N$  is a normal subgroup of a group  $G$ , and  $M$  is a characteristic subgroup of  $N$ , then  $M$  is a normal subgroup of  $G$ .

1.3 Let  $G$  be a group and let  $H$  and  $K$  be subgroups of  $G$ . The *commutator subgroup*  $(H, K)$  is defined as the smallest subgroup containing all elements of the form  $hkh^{-1}k^{-1}$ , where  $h \in H$  and  $k \in K$ . Pick out the true statement(s):

- (a) If  $H$  and  $K$  are normal subgroups, then  $(H, K)$  is a normal subgroup.
- (b) If  $H$  and  $K$  are characteristic subgroups, then  $(H, K)$  is a characteristic subgroup.
- (c)  $(G, G)$  is normal in  $G$  and  $G/(G, G)$  is abelian.

1.4 Write the following permutation as a product of disjoint cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}.$$

1.5 Pick out the true statement(s):

- (a) The set of all  $2 \times 2$  matrices with rational entries (with the usual operations of matrix addition and matrix multiplication) is a ring which has no non-trivial ideals.
- (b) Let  $R = \mathcal{C}[0, 1]$  be considered as a ring with the usual operations of pointwise addition and pointwise multiplication. Let

$$\mathcal{I} = \{f : [0, 1] \rightarrow \mathbb{R} \mid f(1/2) = 0\}.$$

Then  $\mathcal{I}$  is a maximal ideal.

- (c) Let  $R$  be a commutative ring and let  $\mathcal{P}$  be a prime ideal of  $R$ . Then  $R/\mathcal{P}$  is an integral domain.

1.6 What is the degree of the following numbers over  $\mathbb{Q}$ ?

- (a)  $\sqrt{2} + \sqrt{3}$
- (b)  $\sqrt{2}\sqrt{3}$

**1.7** Let  $V$  be the real vector space of all polynomials of degree  $\leq 3$  with real coefficients. Define the linear transformation

$$T(\alpha_0 + \alpha_1x + \alpha_2x^2 + \alpha_3x^3) = \alpha_0 + \alpha_1(x + 1) + \alpha_2(x + 1)^2 + \alpha_3(x + 1)^3.$$

Write down the matrix of  $T$  with respect to the basis  $\{1, x, x^2, x^3\}$  of  $V$ .

**1.8** Let  $A$  be an  $n \times n$  upper triangular matrix with complex entries. Pick out the true statement(s):

- (a) If  $A \neq 0$ , and if  $a_{ii} = 0$ , for all  $1 \leq i \leq n$ , then  $A^n = 0$ .
- (b) If  $A \neq I$  and if  $a_{ii} = 1$  for all  $1 \leq i \leq n$ , then  $A$  is not diagonalizable.
- (c) If  $A \neq 0$ , then  $A$  is invertible.

**1.9** Pick out the true statement(s):

- (a) There exist  $n \times n$  matrices  $A$  and  $B$  with real entries such that

$$(I - (AB - BA))^n = 0.$$

- (b) If  $A$  is a symmetric and positive definite  $n \times n$  matrix, then

$$(\operatorname{tr}(A))^n \geq n^n \det(A)$$

where ‘tr’ denotes the trace and ‘det’ denotes the determinant of a matrix.

- (c) Let  $A$  be a  $5 \times 5$  skew-symmetric matrix with real entries. Then  $A$  is singular.

**1.10** Let  $A$  be a  $5 \times 5$  matrix whose characteristic polynomial is given by

$$(\lambda - 2)^3(\lambda + 2)^2.$$

If  $A$  is diagonalizable, find  $\alpha$  and  $\beta$  such that

$$A^{-1} = \alpha A + \beta I.$$

## Section 2: Analysis

**2.1** Let  $\{a_n\}$  be a sequence of positive real numbers such that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r < 1.$$

Can we evaluate  $\lim_{n \rightarrow \infty} a_n$ ? If 'yes', right down that limit.

**2.2** Test the following series for convergence:

(a)

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+\frac{5}{4}}}.$$

(b)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right).$$

**2.3** Consider the polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

with real coefficients. Pick out the case(s) which ensure that the polynomial  $p(\cdot)$  has a root in the interval  $[0, 1]$ .

(a)  $a_0 < 0$  and  $a_0 + a_1 + \cdots + a_n > 0$ .

(b)

$$a_0 + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0.$$

(c)

$$\frac{a_0}{1.2} + \frac{a_1}{2.3} + \cdots + \frac{a_n}{(n+1)(n+2)} = 0.$$

**2.4** Pick out the true statement(s):

(a) The function

$$f(x) = \frac{\sin(x^2)}{\sin^2 x}$$

is uniformly continuous on the interval  $]0, 1[$ .

(b) A continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is uniformly continuous if it maps Cauchy sequences into Cauchy sequences.

(c) If a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is uniformly continuous, then it maps Cauchy sequences into Cauchy sequences.

**2.5** Test the following for uniform convergence:

(a) The sequence of functions

$$\left\{ \frac{x^n}{1+x^n} \right\}$$

over the interval  $[0, 2]$ .

(b) The series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2 + 1}$$

over  $\mathbb{R}$ .

(c) The sequence of functions

$$\{n^2 x^2 e^{-nx}\}$$

over the interval  $]0, \infty[$ .

**2.6** Evaluate:

$$\lim_{n \rightarrow \infty} \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \cdots \left(1 + \frac{n}{n}\right) \right\}^{\frac{1}{n}}.$$

**2.7** Let  $f : [-\pi, \pi] \rightarrow \mathbb{R}$  be continuous. Pick out the case(s) which imply that  $f \equiv 0$ .

(a)

$$\int_{-\pi}^{\pi} x^n f(x) dx = 0, \text{ for all } n \geq 0.$$

(b)

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = 0, \text{ for all } n \geq 0.$$

(c)

$$\int_{-\pi}^{\pi} f(x) \sin nx dx = 0, \text{ for all } n \geq 1.$$

**2.8** Evaluate:

$$\int_{\Gamma} \frac{dz}{(z^2 + 4)^2}$$

where  $\Gamma = \{z \in \mathbb{C} \mid |z - i| = 2\}$ , described in the anticlockwise (*i.e.* positive) direction.

**2.9** Find the residue at  $z = 1$  of the function:

$$f(z) = \frac{5z - 2}{z(z - 1)}.$$

**2.10** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be analytic. Which of the following conditions imply that  $f$  is a constant function?

(a)  $\operatorname{Re} f(z) > 0$  for all  $z \in \mathbb{C}$ .

(b)  $|f(z)| \in \mathbb{Z}$  for all  $z \in \mathbb{C}$ .

(c)  $f(z) = i$  when  $z = \left(1 + \frac{k}{n}\right) + i$  for every positive integer  $k$ .

### Section 3: Topology

**3.1** Let  $S^1$  denote the unit circle in the plane  $\mathbb{R}^2$ . Pick out the true statement(s):

- (a) There exists  $f : S^1 \rightarrow \mathbb{R}$  which is continuous and one-one.
- (b) For every continuous function  $f : S^1 \rightarrow \mathbb{R}$ , there exist uncountably many pairs of distinct points  $x$  and  $y$  in  $S^1$  such that  $f(x) = f(y)$ .
- (c) There exists  $f : S^1 \rightarrow \mathbb{R}$  which is continuous and one-one and onto.

**3.2** Which of the following metric spaces are separable?

- (a)  $C[0, 1]$  with its usual 'sup-norm' topology.
- (b) The space  $\ell^\infty$  of all bounded real sequences with the metric

$$d(x, y) = \sup_n |x_n - y_n|,$$

where  $x = (x_n)$  and  $y = (y_n)$ .

- (c) The space  $\ell^2$  of all square summable real sequences with the metric

$$d(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^2 \right)^{\frac{1}{2}},$$

where  $x = (x_n)$  and  $y = (y_n)$ .

**3.3** Which of the following sets are nowhere dense?

- (a) The Cantor set in  $[0, 1]$ .
- (b) The  $xy$ -plane in  $\mathbb{R}^3$ .
- (c) Any countable set in  $\mathbb{R}$ .

**3.4** Pick out the true statement(s).

- (a) If  $f : ]-1, 1[ \rightarrow \mathbb{R}$  is bounded and continuous, it is uniformly continuous.
- (b) If  $f : S^1 \rightarrow \mathbb{R}$  is continuous, it is uniformly continuous.
- (c) If  $(X, d)$  is a metric space and  $A \subset X$ , then the function  $f(x) = d(x, A)$  defined by

$$d(x, A) = \inf_{y \in A} d(x, y)$$

is uniformly continuous.

**3.5** Which of the following maps define a homeomorphism?

- (a)  $f : \mathbb{R} \rightarrow ]0, \infty[$ , where  $f(x) = e^x$ .
- (b)  $f : [0, 1] \rightarrow S^1$ , where  $f(t) = (\cos 2\pi t, \sin 2\pi t)$ .
- (c) Any map  $f : X \rightarrow Y$  which is continuous, one-one and onto, if  $X$  is compact and  $Y$  is Hausdorff.

**3.6** Consider the set of all  $n \times n$  matrices with real entries as the space  $\mathbb{R}^{n^2}$ . Which of the following sets are compact?

- (a) The set of all orthogonal matrices.
- (b) The set of all matrices with determinant equal to unity.
- (c) The set of all invertible matrices.

**3.7** In the set of all  $n \times n$  matrices with real entries, considered as the space  $\mathbb{R}^{n^2}$ , which of the following sets are connected?

- (a) The set of all orthogonal matrices.
- (b) The set of all matrices with trace equal to unity.
- (c) The set of all symmetric and positive definite matrices.

**3.8** Let  $X$  be an arbitrary topological space. Pick out the true statement(s):

- (a) If  $X$  is compact, then every sequence in  $X$  has a convergent subsequence.
- (b) If every sequence in  $X$  has a convergent subsequence, then  $X$  is compact.
- (c)  $X$  is compact if, and only if, every sequence in  $X$  has a convergent subsequence.

**3.9** Which of the following metric spaces are complete?

- (a) The space  $\mathcal{C}^1[0, 1]$  of continuously differentiable real-valued functions on  $[0, 1]$  with the metric

$$d(f, g) = \max_{t \in [0, 1]} |f(t) - g(t)|.$$

- (b) The space of all polynomials in a single variable with real coefficients, with the same metric as above.
- (c) The space  $\mathcal{C}[0, 1]$  with the metric

$$d(f, g) = \int_0^1 |f(t) - g(t)| dt.$$

**3.10** Classify the following alphabets into homeomorphism classes:

**N, B, H, M**



## Section 4: Applied Mathematics

**4.1** A body, falling under gravity, experiences a resisting force of air proportional to the square of the velocity of the body. Write down the differential equation governing the motion satisfied by the distance  $y(t)$  travelled by the body in time  $t$ .

**4.2** Reduce the following differential equation to a linear system of first order equations:

$$\frac{d^2x}{dt^2} + P(t)\frac{dx}{dt} + Q(t)x = 0.$$

**4.3** The volume of the unit ball in  $\mathbb{R}^N$  is given by

$$\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2} + 1)}$$

where  $\Gamma(\cdot)$  denotes the usual gamma function. Write down the explicit value of  $\omega_5$ .

**4.4** Consider the differential equation

$$(1+x)y' = py$$

where  $p$  is a constant. Assume that the equation has a power series solution  $y = \sum_{n=0}^{\infty} a_n x^n$ . Write down the recurrence relation for the coefficients  $a_n$ .

**4.5** In the above problem, if  $y(0) = 1$ , use the above series to find a closed form solution to the equation.

**4.6** Classify the following partial differential operators as elliptic, parabolic or hyperbolic:

- (a)  $5u_{xx} + 6u_{xy} + 2u_{yy}$ .
- (b)  $2u_{xx} + 6u_{xy} + 2u_{yy}$ .

**4.7** Let  $f$  and  $g$  be two smooth scalar valued functions. Compute

$$\operatorname{div}(\nabla f \times \nabla g).$$

**4.8** Let  $S$  denote the sphere centred at the origin and of radius  $a > 0$  in  $\mathbb{R}^3$ . Write down the coordinates of the unit outward normal to  $S$  at the point  $(x, y, z) \in S$ .

**4.9** Use Gauss' divergence theorem to evaluate

$$\int \int_S (x^4 + y^4 + z^4) dS$$

where  $S$  is the sphere mentioned in the preceding problem.

**4.10** Consider the domain  $[0, 1] \times [0, T]$ . Let  $h > 0$  and  $k > 0$ . Let  $x_n = nh$  and  $t_m = mk$  for positive integers  $m$  and  $n$ . Let  $u_n^m = u(x_n, t_m)$ . Write down the partial differential equation for which the following discretization defines a numerical scheme:

$$\frac{u_n^{m+1} - u_n^m}{k} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{h^2}.$$

## Section 5: Miscellaneous

**5.1** Let  $n$  be a fixed positive integer and let  $C_k$  denote the usual binomial coefficient  ${}^n C_k$ , the number of ways of choosing  $k$  objects from  $n$  objects. Evaluate:

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}.$$

**5.2** Find the number of ways  $2n$  persons can be seated at 2 round tables, with  $n$  persons at each table.

**5.3** Let a point  $(x, y)$  be chosen at random in the square  $[0, 1] \times [0, 1]$ . Find the probability that  $y \geq x^2$ .

**5.4** Pick out the true statement(s):

- (a) If  $n$  is an odd positive integer, then 8 divides  $n^2 - 1$ .
- (b) If  $n$  and  $m$  are odd positive integers, then  $n^2 + m^2$  is not a perfect square.
- (c) For every positive integer  $n$ ,

$$\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$$

is an integer.

**5.5** Consider a circle of unit radius centered at  $O$  in the plane. Let  $AB$  be a chord which makes an angle  $\theta$  with the tangent to the circle at  $A$ . Find the area of the triangle  $OAB$ .

**5.6** Evaluate:

$$\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \cdots$$

**5.7** Evaluate:

$$1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots$$

**5.8** Find the sum to  $n$  terms as well as the sum to infinity of the series:

$$\frac{1}{3} + \frac{1}{4} \cdot \frac{1}{2!} + \frac{1}{5} \cdot \frac{1}{3!} + \cdots$$

**5.9** If  $a, b$  and  $c$  are all distinct real numbers, find the condition that the following determinant vanishes:

$$\begin{vmatrix} a & a^2 & 1 + a^3 \\ b & b^2 & 1 + b^3 \\ c & c^2 & 1 + c^3 \end{vmatrix}.$$

**5.10** Assume that the line segment  $[0, 2]$  in the  $x$ -axis of the plane acts as a mirror. A light ray from the point  $(0, 1)$  gets reflected off this mirror and reaches the point  $(2, 2)$ . Find the point of incidence on the mirror.